

### 596(1) : Precise Meaning of the Tensorial Taylor Series

In tensorial notation the series is:

$$\Delta f = f(r + \delta r) - f(r) = \frac{\partial f}{\partial r^j} (\delta r)^j + \frac{1}{2!} \frac{\partial^2 f}{\partial r^j \partial r^k} (\delta r)^j (\delta r)^k + \frac{1}{3!} \frac{\partial^3 f}{\partial r^j \partial r^k \partial r^l} (\delta r)^j (\delta r)^k (\delta r)^l + \frac{1}{4!} \frac{\partial^4 f}{\partial r^j \partial r^k \partial r^l \partial r^m} (\delta r)^j (\delta r)^k (\delta r)^l (\delta r)^m + \dots \quad (1)$$

Note carefully that there is summation over repeated indices.

First Term

$$\Delta f^{(1)} = \frac{\partial f}{\partial r^1} (\delta r)^1 + \frac{\partial f}{\partial r^2} (\delta r)^2 + \frac{\partial f}{\partial r^3} (\delta r)^3$$
$$= \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + \delta z \frac{\partial f}{\partial z} \quad (2)$$
$$= \underline{\delta r} \cdot \underline{\nabla} f$$

where

$$\underline{\delta r} = \delta x \underline{i} + \delta y \underline{j} + \delta z \underline{k} \quad (3)$$

and

$$\underline{\nabla} f = \frac{\partial f}{\partial x} \underline{i} + \frac{\partial f}{\partial y} \underline{j} + \frac{\partial f}{\partial z} \underline{k} \quad (4)$$

Second Term

$$\Delta f^{(2)} = \frac{1}{2!} \left[ \frac{\partial^2 f}{\partial r^1 \partial r^k} (\delta r)^1 (\delta r)^k + \frac{\partial^2 f}{\partial r^2 \partial r^k} (\delta r)^2 (\delta r)^k + \frac{\partial^2 f}{\partial r^3 \partial r^k} (\delta r)^3 (\delta r)^k \right] \quad (5)$$

2) Now sum over the kr index to give:

$$\begin{aligned}
 \Delta g^{(2)} &= \frac{1}{2!} \left[ \frac{\partial^2 g}{\partial r^1 \partial r^1} (s_r)' (s_r)' + \frac{\partial^2 g}{\partial r^1 \partial r^2} (s_r)' (s_r)^2 + \frac{\partial^2 g}{\partial r^1 \partial r^3} (s_r)' (s_r)^3 \right. \\
 &\quad + \frac{\partial^2 g}{\partial r^2 \partial r^1} (s_r)^2 (s_r)' + \frac{\partial^2 g}{\partial r^2 \partial r^2} (s_r)^2 (s_r)^2 + \frac{\partial^2 g}{\partial r^2 \partial r^3} (s_r)^2 (s_r)^3 \\
 &\quad \left. + \frac{\partial^2 g}{\partial r^3 \partial r^1} (s_r)^3 (s_r)' + \frac{\partial^2 g}{\partial r^3 \partial r^2} (s_r)^3 (s_r)^2 + \frac{\partial^2 g}{\partial r^3 \partial r^3} (s_r)^3 (s_r)^3 \right] \\
 &= \frac{1}{2!} \left[ (s_x)^2 \frac{\partial^2 g}{\partial x^2} + (s_x)(s_y) \frac{\partial^2 g}{\partial x \partial y} + (s_x)(s_z) \frac{\partial^2 g}{\partial x \partial z} \right. \\
 &\quad + (s_y)(s_x) \frac{\partial^2 g}{\partial y \partial x} + (s_y)^2 \frac{\partial^2 g}{\partial y^2} + (s_y)(s_z) \frac{\partial^2 g}{\partial y \partial z} \\
 &\quad \left. + (s_z)(s_x) \frac{\partial^2 g}{\partial z \partial x} + (s_z)(s_y) \frac{\partial^2 g}{\partial z \partial y} + (s_z)^2 \frac{\partial^2 g}{\partial z^2} \right] \\
 &= \left( s_x \frac{\partial}{\partial x} + s_y \frac{\partial}{\partial y} + s_z \frac{\partial}{\partial z} \right) \left( s_x \frac{\partial g}{\partial x} + s_y \frac{\partial g}{\partial y} + s_z \frac{\partial g}{\partial z} \right) \\
 &= (\underline{s}_r \cdot \underline{\nabla}) (\underline{s}_r \cdot \underline{\nabla} g) \\
 &= (\underline{s}_r \cdot \underline{\nabla}) ((\underline{s}_r \cdot \underline{\nabla}) g) \quad \text{--- (6)} \\
 &= (\underline{s}_r \cdot \underline{\nabla})^2 g
 \end{aligned}$$

The notation  $(\underline{s}_r \cdot \underline{\nabla})$  means  $\underline{s}_r \cdot \underline{\nabla}$  operating on  $g$  !!.

Having checked that the tensorial Taylor expansion (1) gives the vector Taylor expansion:

$$f(\underline{r} + \underline{\delta r}) - f(\underline{r}) = (\underline{\delta r} \cdot \underline{\nabla}) f(\underline{r}) + \frac{1}{2!} (\underline{\delta r} \cdot \underline{\nabla})^2 f(\underline{r}) + \frac{1}{3!} (\underline{\delta r} \cdot \underline{\nabla})^3 f(\underline{r}) + \frac{1}{4!} (\underline{\delta r} \cdot \underline{\nabla})^4 f(\underline{r}) - (7)$$

It is now possible to proceed to the evaluation of higher order terms with the help of computer algebra, and then to apply isotropic averaging.

The result is a general and powerful method of calculating the effect of the vacuum on any function f.

The clearest way of applying isotropic averaging is to use the Cartesian component results:

$$\langle \delta x \rangle = \langle \delta y \rangle = \langle \delta z \rangle = 0, \quad (8)$$

$$\langle \delta x \delta y \rangle = \langle \delta x \delta z \rangle = \langle \delta y \delta z \rangle = 0, \quad (9)$$

$$\langle \delta x^2 \rangle = \langle \delta y^2 \rangle = \langle \delta z^2 \rangle = \frac{1}{3} \langle \underline{\delta r} \cdot \underline{\delta r} \rangle \quad (10)$$

It follows that:

$$\begin{aligned} \langle \Delta f^{(1)} \rangle &= \langle \delta x \rangle \frac{\partial f}{\partial x} + \langle \delta y \rangle \frac{\partial f}{\partial y} + \langle \delta z \rangle \frac{\partial f}{\partial z} \\ &= 0 \end{aligned} \quad (11)$$

$$\begin{aligned} \langle \Delta f^{(2)} \rangle &= \frac{1}{2!} \left( \langle \delta x^2 \rangle \frac{\partial^2 f}{\partial x^2} + \langle \delta y^2 \rangle \frac{\partial^2 f}{\partial y^2} + \langle \delta z^2 \rangle \frac{\partial^2 f}{\partial z^2} \right) \\ &= \frac{1}{6} \langle \underline{\delta r} \cdot \underline{\delta r} \rangle \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \\ &= \frac{1}{6} \langle \underline{\delta r} \cdot \underline{\delta r} \rangle \nabla^2 f \end{aligned} \quad (12)$$

This is a very useful and important result because the effect of the vacuum on any scalar function  $f$  can be calculated by finding  $\langle \underline{S} \cdot \underline{S} \rangle$  for the vacuum. If  $f$  is the Coulomb potential, eq. (12) gives a precise expansion of the Landau shift.

The purpose of UFT 396 is to calculate the higher order terms in the Taylor series:

$$\langle \Delta f \rangle = \frac{1}{6} \langle \underline{S} \cdot \underline{S} \rangle \nabla^2 f + \dots - (13)$$

and to apply the method to physics in general. The meaning of the condensed notation (7) is by no means clear, and tensor notation as in eq. (1) is not used by anyone but a minority of physicists. Therefore component notation is by far the clearest. It follows that:

$$\begin{aligned} \Delta f &= S_x \frac{\partial f}{\partial x} + S_y \frac{\partial f}{\partial y} + S_z \frac{\partial f}{\partial z} \\ &+ \frac{1}{2!} \left( S_x \frac{\partial}{\partial x} + S_y \frac{\partial}{\partial y} + S_z \frac{\partial}{\partial z} \right) \left( S_x \frac{\partial f}{\partial x} + S_y \frac{\partial f}{\partial y} + S_z \frac{\partial f}{\partial z} \right) \\ &+ \frac{1}{3!} \left( S_x \frac{\partial}{\partial x} + S_y \frac{\partial}{\partial y} + S_z \frac{\partial}{\partial z} \right) \left[ \left( S_x \frac{\partial}{\partial x} + S_y \frac{\partial}{\partial y} + S_z \frac{\partial}{\partial z} \right) \left( S_x \frac{\partial f}{\partial x} + S_y \frac{\partial f}{\partial y} + S_z \frac{\partial f}{\partial z} \right) \right] \\ &+ \frac{1}{4!} \left( S_x \frac{\partial}{\partial x} + S_y \frac{\partial}{\partial y} + S_z \frac{\partial}{\partial z} \right) \left[ \left( S_x \frac{\partial}{\partial x} + S_y \frac{\partial}{\partial y} + S_z \frac{\partial}{\partial z} \right) \left[ \left( S_x \frac{\partial}{\partial x} + S_y \frac{\partial}{\partial y} + S_z \frac{\partial}{\partial z} \right) \left( S_x \frac{\partial f}{\partial x} + S_y \frac{\partial f}{\partial y} + S_z \frac{\partial f}{\partial z} \right) \right] \right] \\ &+ \dots \end{aligned} - (14)$$

Eq. (14) can be worked out with computer algebra.